

Spin-Hamilton Operator, Graviton-Photon Coupling and an Eigenvalue Problem

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Abstract We solve exactly the eigenvalue problem for a spin Hamilton operator describing graviton-photon coupling. Entanglement of the eigenstates are also studied.

1 Introduction

The photon γ is a spin-1 particle without rest mass and is described by the traceless hermitian 3×3 spin matrices

$$p_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad p_2 = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad p_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

with $x \leftrightarrow 1$, $y \leftrightarrow 2$, $z \leftrightarrow 3$. The eigenvalues of these matrices are $+1$, 0 , -1 . The normalized eigenvectors for p_1 are

$$\mathbf{u}_{1,1} = \frac{1}{2} \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}, \quad \mathbf{u}_{1,0} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \mathbf{u}_{1,-1} = \frac{1}{2} \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}.$$

The normalized eigenvectors for p_2 are

$$\mathbf{u}_{2,1} = \frac{1}{2} \begin{pmatrix} 1 \\ \sqrt{2}i \\ -1 \end{pmatrix}, \quad \mathbf{u}_{2,0} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{u}_{2,-1} = \frac{1}{2} \begin{pmatrix} 1 \\ -\sqrt{2}i \\ -1 \end{pmatrix}.$$

The normalized eigenvectors for p_3 is the standard basis denoted by $\mathbf{u}_{3,1}$, $\mathbf{u}_{3,0}$, $\mathbf{u}_{3,-1}$. We have the commutation relations

$$[p_1, p_2] = ip_3, \quad [p_2, p_3] = ip_1, \quad [p_3, p_1] = ip_2.$$

The graviton g is a spin-2 particle without rest mass and described by the traceless hermitian 5×5 spin matrices

$$g_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & \sqrt{6}/2 & 0 & 0 \\ 0 & \sqrt{6}/2 & 0 & \sqrt{6}/2 & 0 \\ 0 & 0 & \sqrt{6}/2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad g_2 = i \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & -\sqrt{6}/2 & 0 & 0 \\ 0 & \sqrt{6}/2 & 0 & -\sqrt{6}/2 & 0 \\ 0 & 0 & \sqrt{6}/2 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$g_3 = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{pmatrix}.$$

The eigenvalues of these matrices are $+2, +1, 0, -1, -2$. The normalized eigenvectors for g_1 are

$$\mathbf{v}_{1,-2} = \frac{1}{4} \begin{pmatrix} 1 \\ -2 \\ \sqrt{6} \\ -2 \\ 1 \end{pmatrix}, \quad \mathbf{v}_{1,2} = \frac{1}{4} \begin{pmatrix} 1 \\ 2 \\ \sqrt{6} \\ 2 \\ 1 \end{pmatrix},$$

$$\mathbf{v}_{1,-1} = \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \\ -1 \end{pmatrix}, \quad \mathbf{v}_{1,1} = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \\ -1 \end{pmatrix}, \quad \mathbf{v}_{1,0} = \frac{\sqrt{3}}{\sqrt{8}} \begin{pmatrix} 1 \\ 0 \\ -\sqrt{2/3} \\ 0 \\ 1 \end{pmatrix}.$$

The normalized eigenvectors for g_2 are

$$\mathbf{v}_{2,-2} = \frac{1}{4} \begin{pmatrix} 1 \\ -2i \\ -\sqrt{6} \\ 2i \\ 1 \end{pmatrix}, \quad \mathbf{v}_{2,2} = \frac{1}{4} \begin{pmatrix} 1 \\ 2i \\ -\sqrt{6} \\ -2i \\ 1 \end{pmatrix},$$

$$\mathbf{v}_{2,-1} = \frac{1}{2} \begin{pmatrix} 1 \\ -i \\ 0 \\ -i \\ -1 \end{pmatrix}, \quad \mathbf{v}_{2,1} = \frac{1}{2} \begin{pmatrix} 1 \\ i \\ 0 \\ i \\ -1 \end{pmatrix}, \quad \mathbf{v}_{2,0} = \frac{\sqrt{3}}{\sqrt{8}} \begin{pmatrix} 1 \\ 0 \\ \sqrt{2/3} \\ 0 \\ 1 \end{pmatrix}.$$

The normalized eigenvectors for g_3 are the standard basis \mathbf{e}_j ($j = 1, \dots, 5$) with $\mathbf{v}_{3,2} = \mathbf{e}_1$, $\mathbf{v}_{3,1} = \mathbf{e}_2$, $\mathbf{v}_{3,0} = \mathbf{e}_3$, $\mathbf{v}_{3,-1} = \mathbf{e}_4$, $\mathbf{v}_{3,-2} = \mathbf{e}_5$. We have the commutation

relations

$$[g_1, g_2] = ig_3, \quad [g_2, g_3] = ig_1, \quad [g_3, g_1] = ig_2.$$

We investigate the eigenvalue problem for the Hamilton operator of the coupled photon-graviton spin system

$$\hat{K} \equiv \frac{\hat{H}}{\hbar\omega} = p_1 \otimes g_1 \otimes p_1 + p_2 \otimes g_2 \otimes p_2 + p_3 \otimes g_3 \otimes p_3$$

where \otimes denotes the Kronecker product [1]. The Hamilton operator \hat{K} acts in the Hilbert space \mathbb{C}^{45} . Note that \hat{K} is a hermitian matrix and thus the eigenvalues are real. Since the trace of the matrices $p_1, p_2, p_3, g_1, g_2, g_3$ vanish, we find that $\text{tr}(\hat{K}) = 0$. Consequently the sum of the 45 eigenvalues is 0. We also study the entanglement of the non-degenerate eigenvectors utilizing the Schmidt decomposition [2, 3, 4, 5, 6, 7].

2 Spectrum

To find an estimate for the eigenvalue we can utilize the inequality

$$|\lambda| \leq \max_{1 \leq j \leq n} \sum_{\ell=1}^n |a_{j\ell}|$$

which is valid for any eigenvalue of an $n \times n$ matrix A . For the Hamilton operator $\hat{K} = (k_{j\ell})$ we find

$$\max_{1 \leq j \leq 45} \sum_{\ell=1}^{45} |k_{j\ell}| = 4\sqrt{3}.$$

A numerical study to find the eigenvalues of the Hamilton operator \hat{K} using the eigenvalue packages of R [8] and Octave [9] provides the hint that 0 (7 times degenerate), 1 and -1 (each 6 times degenerate) and 2 and -2 (each 6 times degenerate) are eigenvalues. Now we calculate symbolically the characteristic polynomial $\det(\hat{K} - \lambda I_{45})$. The eigenvalues given above can now be used to reduce the characteristic polynomial and we finally arrive at the following 45 eigenvalues ordered from smallest to largest

$$\lambda_{1,2,3} = -\frac{\sqrt{\sqrt{33}+9}}{\sqrt{2}}, \quad (3 \text{ times})$$

$$\lambda_{4,5,6,7,8,9} = -2, \quad (6 \text{ times})$$

$$\begin{aligned}
\lambda_{10} &= -\sqrt{3} \text{ (1 times)} \\
\lambda_{11,12,13} &= -\frac{\sqrt{9-\sqrt{33}}}{\sqrt{2}} \text{ (3 times)} \\
\lambda_{14,15,16,17,18,19} &= -1 \text{ (6 times)} \\
\lambda_{20,21,22,23,24,25,26} &= 0 \text{ (7 times)} \\
\lambda_{27,28,29,30,31,32} &= 1 \text{ (6 times)} \\
\lambda_{33,34,35} &= \frac{\sqrt{9-\sqrt{33}}}{\sqrt{2}} \text{ (3 times)} \\
\lambda_{36} &= \sqrt{3} \text{ (1 times)} \\
\lambda_{37,38,39,40,41,42} &= 2, \text{ (6 times)} \\
\lambda_{43,44,45} &= \frac{\sqrt{\sqrt{33}+9}}{\sqrt{2}}, \text{ (3 times)}
\end{aligned}$$

The eigenvalues are symmetric around 0. Only the eigenvalues $\sqrt{3}$ and $-\sqrt{3}$ are not degenerate. The normalized eigenvectors \mathbf{w}_j ($j = 1, \dots, 45$) are pairwise orthogonal and thus form an orthonormal basis in the Hilbert space \mathbb{C}^{45} .

Owing to the degeneracies of most of the eigenvalues the Hamilton operator K admits symmetries, i.e.

$$P^T \hat{K} P = \hat{K}$$

where P is a 45×45 permutation matrix. One of them is the permutation matrix

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \otimes I_5 \otimes \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

where I_5 is the 5×5 identity matrix and the first matrix in the Kronecker product is the 3×3 NOT-gate, i.e. the 3×3 matrix with all 1's in the counter-diagonal and 0 otherwise. We have $U_{NOT} = U_{NOT}^*$ and $\hat{K} = P^* \hat{K} P$.

All symmetries of \hat{K} are given by

$$U \left(\bigoplus_{j=1}^{11} V_j \right) U^*$$

and V_1, V_2, V_3 and V_4 are unitary 6×6 matrices, V_5, V_6, V_7, V_8 are unitary 3×3 matrices, V_9 and V_{10} are 1×1 and unitary and V_{11} is a 7×7 unitary matrix, and

$$\begin{aligned}
U = & \sum_{j=1}^6 \mathbf{w}_{\lambda_4,j} \mathbf{e}_j + \sum_{j=1}^6 \mathbf{w}_{\lambda_{14},j} \mathbf{e}_{j+6} + \sum_{j=1}^6 \mathbf{w}_{\lambda_{27},j} \mathbf{e}_{j+12} + \sum_{j=1}^6 \mathbf{w}_{\lambda_{37},j} \mathbf{e}_{j+18} \\
& + \sum_{j=1}^3 \mathbf{w}_{\lambda_1,j} \mathbf{e}_{j+24} + \sum_{j=1}^3 \mathbf{w}_{\lambda_{11},j} \mathbf{e}_{j+27} + \sum_{j=1}^3 \mathbf{w}_{\lambda_{33},j} \mathbf{e}_{j+30} + \sum_{j=1}^3 \mathbf{w}_{\lambda_{43},j} \mathbf{e}_{j+33} \\
& + \mathbf{w}_{\lambda_{10}} \mathbf{e}_{j+36} + \mathbf{w}_{\lambda_{36}} \mathbf{e}_{j+37} + \sum_{j=1}^7 \mathbf{w}_{\lambda_{20},j} \mathbf{e}_{j+38}
\end{aligned}$$

where $\mathbf{w}_{\lambda_k,j}$ are the elements of an orthonormal basis for each eigenvalue λ_k of \hat{K} , and $\{\mathbf{e}_1, \dots, \mathbf{e}_{45}\}$ denotes the standard basis in \mathbb{C}^{45} .

3 Entanglement

The normalized eigenvectors of \hat{K} are elements of the Hilbert space \mathbb{C}^{45} and form an orthonormal basis in \mathbb{C}^{45} . Now $45 = 9 \cdot 5 = 3 \cdot 5 \cdot 3$. The normalized eigenvectors are pairwise orthogonal and form an orthonormal basis in \mathbb{C}^{45} . We ask the question whether the eigenvectors can be written as the Kronecker product of vectors in \mathbb{C}^9 and vectors in \mathbb{C}^5 . For the vector space \mathbb{C}^9 we could ask again whether the vector can be written as a Kronecker product of two vectors in \mathbb{C}^3 .

Consider the eigenvectors belonging to the non-degenerate eigenvalues $\pm\sqrt{3}$ (page 6). We find the Schmidt decompositions

$$\begin{aligned}
& \sqrt{24(2 - \pm\sqrt{3})} \mathbf{w}_{\pm\sqrt{3}}^T \\
& = P_{GB} (1, 0, 0, 0, 0, 0, 0, 0, 1) \otimes (0, 1, 0, (\pm\sqrt{3} - 2)i, 0) \\
& \quad + P_{GB} (0, 0, 1, 0, 0, 0, 1, 0, 0) \otimes (0, (2 - \pm\sqrt{3})i, 0, -1, 0) \\
& \quad + (1 - \pm\sqrt{3})(i + 1) P_{GB} (0, 0, 0, 0, 1, 0, 0, 0, 0) \otimes (-1, 0, 0, 0, 1)
\end{aligned}$$

(where P_{GB} is the permutation matrix which rearranges the photon-photon-graviton state as a photon-graviton-photon state) which expresses the (photon pair) – (graviton) entanglement,

$$\mathbf{w}_{\sqrt{3}} = \frac{1}{\sqrt{24(2-\sqrt{3})}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ (2-\sqrt{3})i \\ 0 \\ 0 \\ 0 \\ (\sqrt{3}-2)i \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ (\sqrt{3}-1)(i+1) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ (1-\sqrt{3})(i+1) \\ 0 \\ 0 \\ 0 \\ 0 \\ (2-\sqrt{3})i \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ (\sqrt{3}-2)i \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{w}_{-\sqrt{3}} = \frac{1}{\sqrt{24(2+\sqrt{3})}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ (\sqrt{3}+2)i \\ 0 \\ 0 \\ 0 \\ -(\sqrt{3}+2)i \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ -(\sqrt{3}+1)(i+1) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ (\sqrt{3}+1)(i+1) \\ 0 \\ 0 \\ 0 \\ 0 \\ (\sqrt{3}+2)i \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ -(\sqrt{3}+2)i \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Eigenvectors for $\sqrt{3}$ and $-\sqrt{3}$.

$$\begin{aligned}
& \sqrt{24(2 - \pm\sqrt{3})} \mathbf{w}_{\pm\sqrt{3}}^T \\
&= (0, (\pm\sqrt{3} - 1)(i + 1), 0) \otimes (0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0) \\
&\quad + (1, 0, (2 - \pm\sqrt{3})i, 0) \otimes (0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0) \\
&\quad + ((2 - \pm\sqrt{3})i, 0, 1) \otimes (0, 0, 0, 0, 0, 1, 0, 0, 0, 0, -1, 0, 0, 0, 0)
\end{aligned}$$

which expresses the (first photon) – (graviton - second photon) entanglement, and

$$\begin{aligned}
& \sqrt{24(2 - \pm\sqrt{3})} \mathbf{w}_{\pm\sqrt{3}}^T \\
&= (0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0) \otimes (1, 0, (2 - \pm\sqrt{3})i) \\
&\quad + (0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0) \otimes ((\pm\sqrt{3} - 2)i, 0, -1) \\
&\quad + (0, 0, 0, 0, 0, 1, 0, 0, 0, 0, -1, 0, 0, 0, 0) \otimes (0, (\pm\sqrt{3} - 1)(1 + i), 0)
\end{aligned}$$

which expresses the (first photon - graviton) – (second photon) entanglement.

4 Conclusion

We considered a spin Hamilton for graviton-photon coupling. The eigenvectors of the associated Hamilton operator, for non-degenerate eigenvalues, yield pairwise entangled systems. Entanglement in the eigenspaces for non-degenerate eigenvalues has not been investigated. Also the dynamics of the entanglement under this Hamilton operator can still be investigated.

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